The delta function well

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- Bound states & scattering states

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- Real potentials

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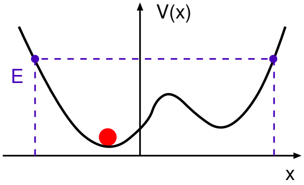

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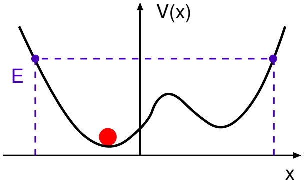

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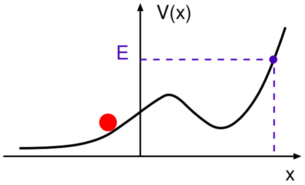

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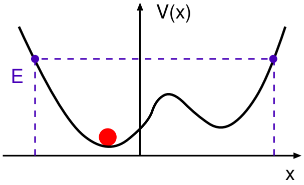

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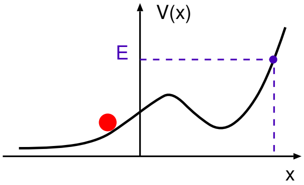

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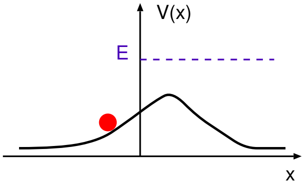

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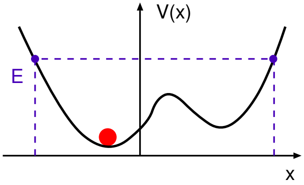

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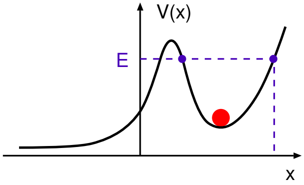

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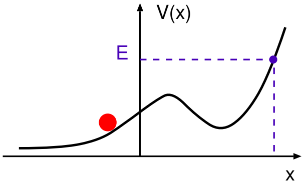

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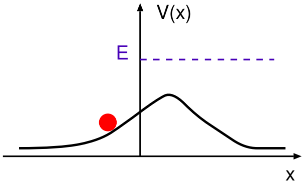

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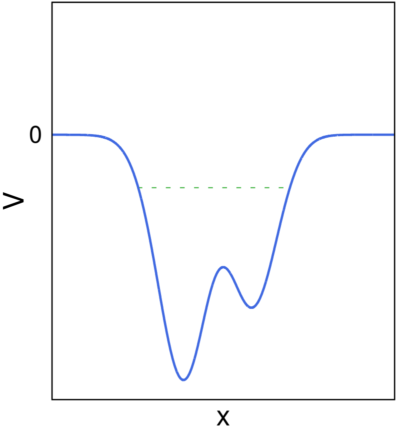

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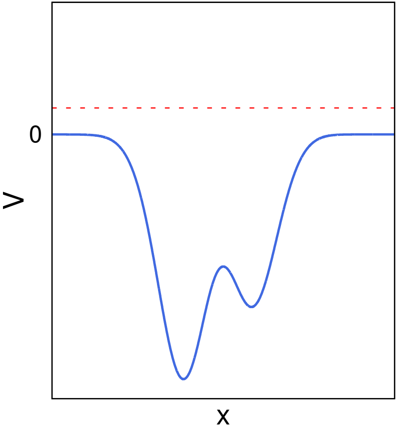

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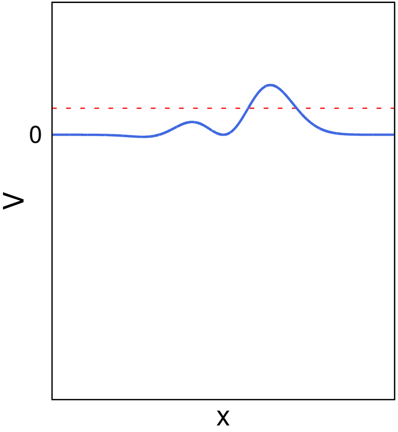

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This integral works for any limits which include the peak of the delta function.

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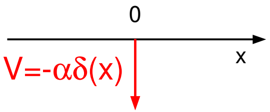

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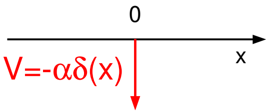

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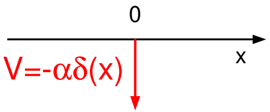

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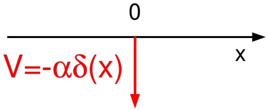

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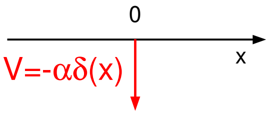

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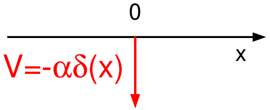

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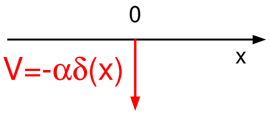

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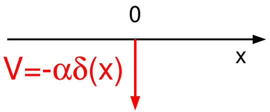

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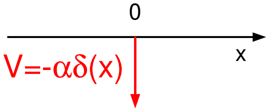

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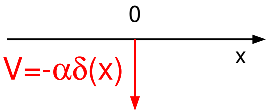

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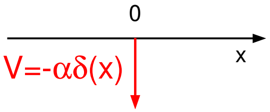

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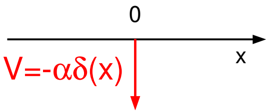

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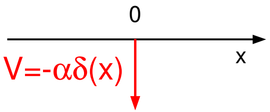

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$$= Be^{+\kappa x}, \quad (x < 0)$$

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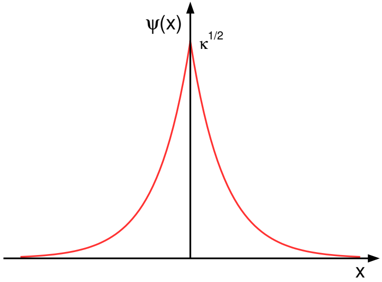

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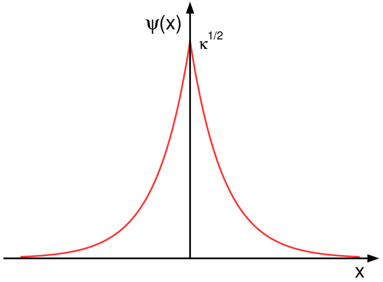

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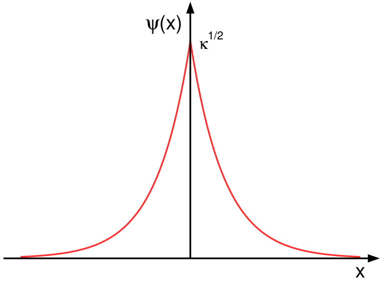

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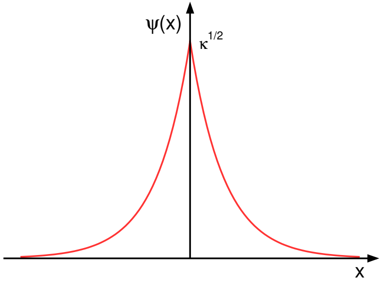

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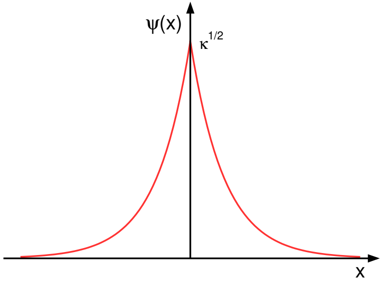

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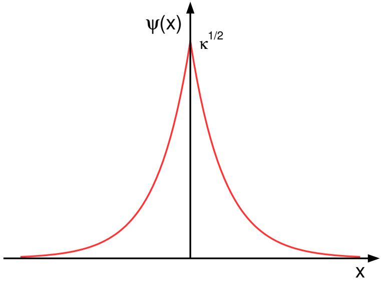

normalizing

$$1 = \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 2|B|^2 \int_0^{\infty} e^{-2\kappa x} dx = \frac{|B|^2}{\kappa} \quad \rightarrow \quad B = \sqrt{\kappa}$$

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The last term is usually zero, unless $V(x) \to \infty$

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For the delta function, this limit is non-zero and can be calculated

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Thus, the negative delta-function potential has a single bound state with wave function and energy. There is always only one bound state for this potential, independent of the strength of the potential α.

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \quad E = -\frac{m\alpha^2}{2\hbar^2}$$

The finite square well

The finite square well

- General solution for three regions

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- General solution for three regions
- Applying the boundary conditions

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- General solution for three regions
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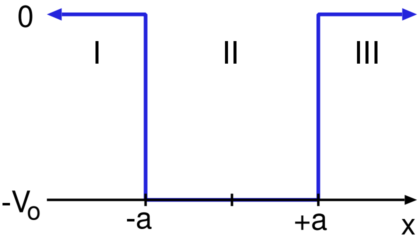
- Limiting cases

Finite square well



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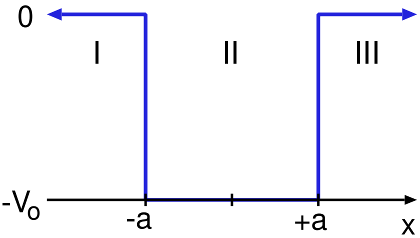
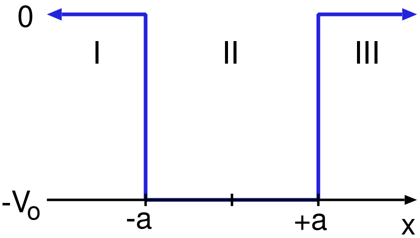


In region I, $x < -a$

Finite square well



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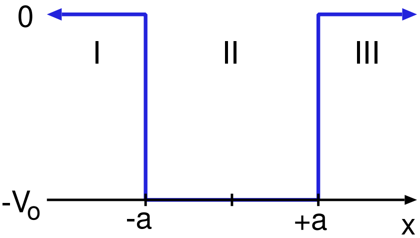

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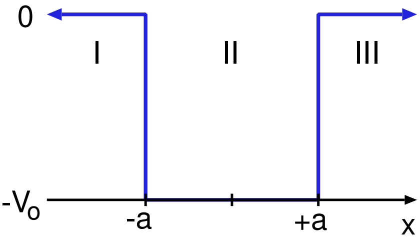

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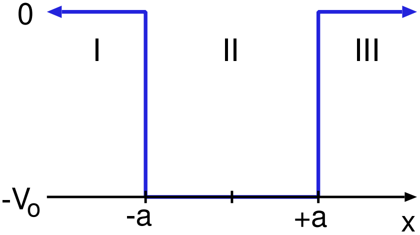

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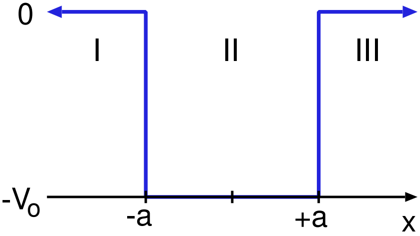

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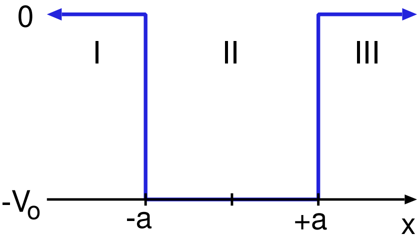

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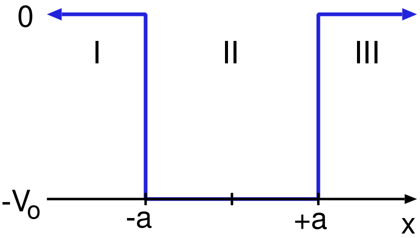

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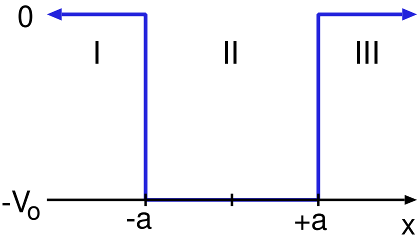

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$$\psi = C\sin(lx) + D\cos(lx)$$

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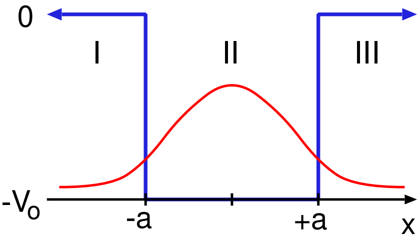

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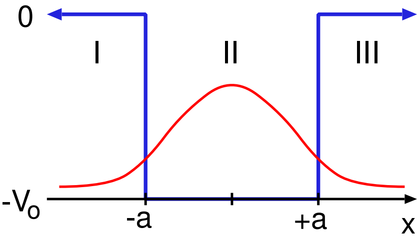

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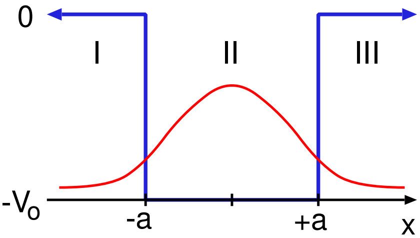

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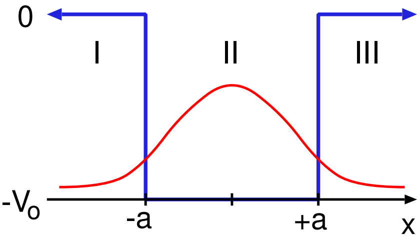

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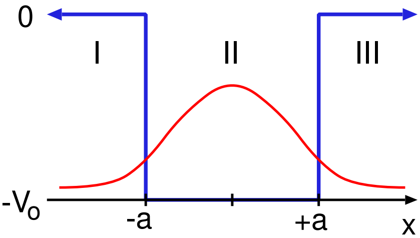

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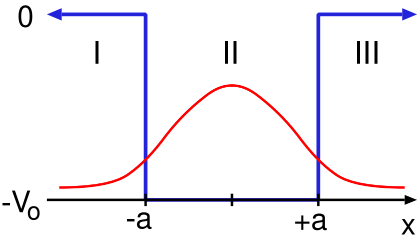

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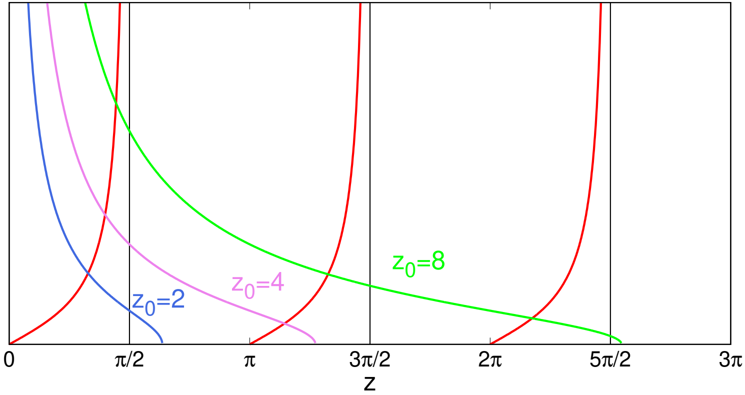

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Since $E_n + V_0$ is just the energy above the bottom of the well and the width is $2a$, the even solutions (and the odd ones, of course) approach those of the infinite square well.

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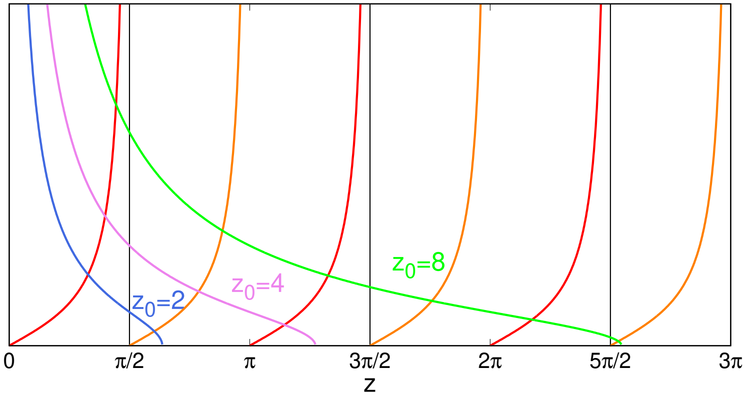

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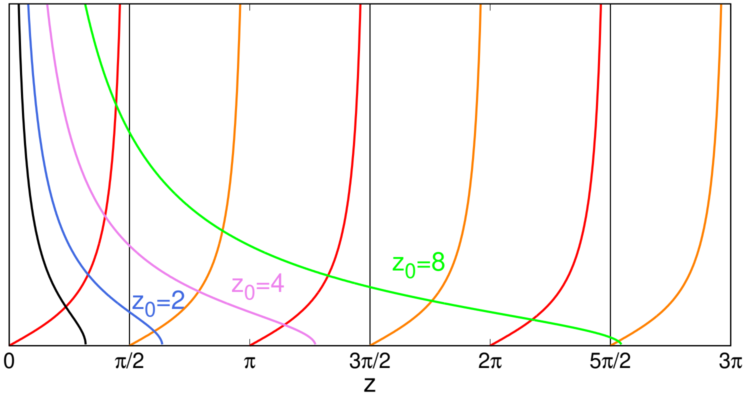

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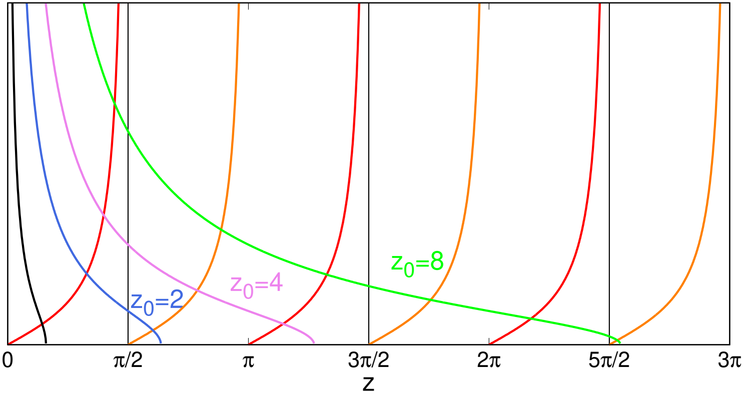

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The number of states decreases until the lowest odd bound state vanishes. However, the ground state (lowest even state) will never vanish. There is *always* a bound state no matter how shallow the well!